Jackson 4.7(a)

$$\Phi(\vec{x}) = \frac{1}{\varepsilon_0} \sum_{l,m} \frac{1}{2l+1} \langle Y_{lm} | r'^{l} | \rho(\vec{x}') \rangle \frac{Y_{lm}(\theta,\phi)}{r^{(l+1)}}$$

$$\rho(\vec{x}') = \frac{1}{64\pi} r'^2 e^{-r'} \sin^2\theta$$ is azimuthally symmetric

$\Rightarrow \langle Y_{lm} | r'^{l} | \rho(\vec{x}') \rangle = 0$ for $m \neq 0$.

$\Rightarrow$ We wish to compute $\langle Y_{l0} | r'^{l} | \rho(\vec{x}') \rangle$.

Recall $Y_{lm} = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m e^{im\phi}$.

$$\Rightarrow \langle Y_{l0} | r'^{l} | \rho(\vec{x}') \rangle$$

$$= \sqrt{\frac{2l+1}{4\pi}} \langle P_l | r'^{l} | \rho(\vec{x}') \rangle.$$

* we have reduced the problem to computing

$$\langle P_l | r'^{l} | \rho(\vec{x}') \rangle.$$

$$\langle P_\ell \,|\, r'^{\ell} \,|\, \rho(\vec{x}') \rangle = \int P_\ell[\cos\theta]\, r'^{\ell} \rho(\vec{x}')\, d^3\vec{x}'$$

$$= \int P_\ell[\cos\theta']\, r'^{\ell} \rho(\theta', r') \sin\theta'\, r'^2\, dr'\, d\theta'\, d\phi'$$

$$= 2\pi \int P_\ell[\cos\theta']\, r'^{\ell} \rho(\theta', r') \sin\theta'\, r'^2\, dr'\, d\theta'$$

$$= \frac{2\pi}{64\pi} \int P_\ell[\cos\theta']\, r'^{\ell}\, r'^2\, e^{-r'} \sin^2\theta' \sin\theta'\, r'^2\, dr'\, d\theta'$$

$$= \frac{1}{32} \int P_\ell[\cos\theta']\, r'^{(\ell+4)} e^{-r'} \sin^3\theta'\, d\theta'\, dr'$$

$$* \quad = \frac{1}{32} \left[\int_0^\infty r'^{(\ell+4)} e^{-r'}\, dr' \right] \left[\int_0^\pi P_\ell[\cos\theta'] \sin^3\theta'\, d\theta' \right]$$

By definition, $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$

$$\Rightarrow \int_0^\infty r'^{(\ell+4)} e^{-r'}\, dr' = \boxed{\Gamma(\ell+5)}$$

$$\int_0^{\pi} P_\ell[\cos\theta'] \sin^3\theta' \, d\theta' \qquad\qquad x \equiv \cos\theta'$$

$$= \int_{-1}^{1} P_\ell[x] \sin^2\theta' \, dx$$

$$* \quad = \int_{-1}^{1} P_\ell[x](1-x^2)\, dx.$$

$1-x^2$ is a linear combination of P_ℓ, because recall

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2-1)$$
$$\vdots$$

we have $1-x^2 = \frac{2}{3}P_0 - \frac{2}{3}P_2$.

$$\Rightarrow \int_{-1}^{1} P_\ell[x](1-x^2)\, dx = \int_{-1}^{1} P_\ell[x]\left(\frac{2}{3}P_0[x] - \frac{2}{3}P_2[x]\right) dx$$

By orthogonality of $\langle P_{\ell'} | P_\ell \rangle = \frac{2}{2\ell+1}\delta_{\ell'\ell}$, we have

$$\int_{-1}^{1} P_\ell[x](1-x^2)\,dx = \begin{cases} \frac{2}{3}\,\frac{2}{2(0)+1} = \frac{4}{3} & \ell = 0 \\ -\frac{2}{3}\,\frac{2}{2(2)+1} = -\frac{4}{15} & \ell = 2. \end{cases}$$

Putting the 2 terms together, we have

$$\langle P_\ell | r'^{\ell} | \rho(\vec{x}') \rangle = \begin{cases} \frac{1}{32}\Gamma(5)\frac{4}{3} & \ell = 0 \\ \frac{1}{32}\Gamma(7)\left(-\frac{4}{15}\right) & \ell = 2. \end{cases}$$

~~This gives~~

$$\cancel{\langle Y_{\ell 0} | r'^{\ell} | \rho(\vec{x}') \rangle =}$$

Going back to $\Phi(\vec{x}) = \frac{1}{\varepsilon_0}\sum_{\ell m}\frac{1}{2\ell+1}\langle \cdots \rangle \frac{Y_{\ell m}(\theta,\phi)}{r^{(\ell+1)}}$

we now know

$$\Phi(\vec{x}) = \frac{1}{\varepsilon_0}\sum_{\ell}\frac{1}{2\ell+1}\sqrt{\frac{2\ell+1}{4\pi}}\langle P_\ell | r'^{\ell} | \rho(\vec{x}') \rangle \sqrt{\frac{2\ell+1}{4\pi}}\frac{P_\ell[\cos\theta]}{r^{(\ell+1)}}$$

$$= \frac{1}{\varepsilon_0 4\pi}\sum_{\ell}\langle P_\ell | r'^{\ell} | \rho(\vec{x}') \rangle \frac{P_\ell[\cos\theta]}{r^{(\ell+1)}}$$

$$= \frac{1}{\cancel{4}\pi\varepsilon_0}\left[\frac{1}{32}\Gamma(5)\frac{\cancel{4}}{3}\frac{P_0[\cos\theta]}{r} - \frac{1}{32}\Gamma(7)\frac{\cancel{4}}{15}\frac{P_2[\cos\theta]}{r^3}\right]$$

$$= \boxed{\frac{1}{32\pi\varepsilon_0}\left[\frac{1}{3}\frac{\Gamma(5)}{r} - \frac{1}{15}\frac{\Gamma(7)}{r^3}\frac{1}{2}\left[3\cos^2\theta - 1\right]\right]}$$

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